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Volume and diameter of a graph and Ollivier's Ricci curvature

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ABSTRACT

We obtain upper bounds of diameter and volume for finite graphs by Ollivier's Ricci curvature.

1. Introduction

In Riemannian geometry, lower bounds on the Ricci curvature allow one to obtain global geometric and topological information such as volume, diameter and fundamental group by comparison with the geometry of a space form. It has been an important problem to generalize the notion of Ricci curvature to general metric-measure spaces. Sturm [9,10], and Lott and Villani [4] introduced a notion of lower bound on Ricci curvature for length spaces equipped with a measure. It relies on ideas from optimal transport theory. But it is difficult to apply to discrete settings.

Ollivier defined a coarse Ricci curvature in a metric space X equipped with random walk m [5]. A random walk m on X is a family of probability measures $m_x(\cdot)$ on X for each $x \in X$ satisfying the following assumptions: (i) the measure m_x depends measurably on the points $x \in X$; (ii) for any $o \in X$, for any $x \in X$ one has $\int d(o, y)dm_x(y) < \infty$.

He defined the coarse Ricci curvature as follows [5].

Definition 1. Let $x, y \in X$ be two distinct points. The coarse Ricci curvature along (xy) is

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}$$

where $W_1(\cdot, \cdot)$ is the L^1 transportation distance.

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The L^1 transportation distance W_1 between two measures is defined as follows:

$$W_1(\nu_1, \nu_2) = \inf_{\xi \in \Pi(\nu_1,\nu_2)} \int_{(x,y)\in X\times X} d(x, y) d\xi(x, y),$$

where $\Pi(\nu_1, \nu_2)$ is the set of measures on $X \times X$ projecting to ν_1 and ν_2 [11].

Ollivier proved a Bonnet–Myers type theorem with positive coarse Ricci curvature, i.e. he obtained an upper bound of diameter with $\kappa(x, y) \geq k > 0$ for any $x, y \in X$ [5]. (See Theorem 5 in Section 2.)

Volume comparison is one of the most important tools to study global structures of Riemannian manifolds with Ricci curvature bounded below. But it is difficult to obtain a volume comparison with Ollivier's Ricci curvature. However in the case of a graph, there are natural random walk m and metric d as follows. Let $G = G(V, E)$ be a finite graph with a set of vertices V and a set of edges E. We define the distance $d(x, y)$ for $x, y \in V$ as the length of the shortest path connecting x and y, i.e. the minimal number of edges connecting x and y. For $x, y \in V$,

$$m_x(y) = \begin{cases} \dfrac{\text{the number of edges connecting } x \text{ and } y}{\deg(x)} & \text{if } y \sim x \text{ and } y \neq x \\ 2 \cdot \dfrac{\text{the number of loops joining } x}{\deg(x)} & \text{if } y = x \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(x)$ is the degree of x and $x \sim y$ if x and y are connected by an edge. Recall that $\deg(x)$ is the number of edges that connect to x, where a loop is counted twice. If G is a simple graph, then

$$m_x(y) = \begin{cases} \dfrac{1}{\deg(x)} & \text{if } y \sim x \\ 0 & \text{otherwise.} \end{cases}$$

We note that if $\kappa(x, y) \geq k$ for any points x, y with $d(x, y) = 1$, then $\kappa(x, y) \geq k$ for any $x, y \in V$ [5]. Let $N(x)$ be the neighborhood of x defined as $\{y \mid x \sim y\}$ and Δ be the maximal degree of a graph G. Let

$$S_p(r) = \{x \in V \mid d(x, p) = r\}$$
$$B_p(r) = \{x \in V \mid d(x, p) \leq r\}.$$

Also we denote the cardinal number of A by $|A|$. With the metric and random walk, we can obtain some results on volume and diameter.

Theorem 1. *Assume that there exists $p \in V$ such that $\kappa(p, x) \geq k$ for all $x \in V$. Then*

$$|S_p(r + 1)| \leq \Delta \left(1 - \frac{k}{2} r\right) |S_p(r)| \tag{1.1}$$

for $k \in \mathbb{R}$. So for $r \geq 1$,

$$|S_p(r)| \leq \Delta^r \prod_{m=0}^{r-1} \left(1 - \frac{k}{2} m\right). \tag{1.2}$$

If we let $C(k, r, \Delta) = 1 + \sum_{n=1}^{r} \left(\Delta^n \prod_{m=0}^{n-1} (1 - \frac{k}{2} m)\right)$, then

$$|B_p(r)| \leq C(k, r, \Delta). \tag{1.3}$$

If $k > 0$, then we obtain various upper bounds of $|B_p(r)|$ from Theorem 1 as follows: since $(1 - \frac{k}{2} m)\Delta \leq (1 - \frac{k}{2})\Delta$ for $m \geq 1$,

$$|B_p(r)| \leq 1 + \Delta + \Delta^2 \left(1 - \frac{k}{2}\right) \left(\frac{\left(\left(1 - \frac{k}{2}\right) \Delta\right)^{r-1} - 1}{\left(1 - \frac{k}{2}\right) \Delta - 1}\right).$$

Let $[x] = \max\{n \mid n \leq x,\ n \in \mathbb{Z}\}$. Assume that $\Delta \geq 3$. (The cases of $\Delta = 1, 2$ are quite trivial.) Since $|S_p(r+1)| \leq (\Delta - 1)|S_p(r)|$ for $1 \leq r \leq [\frac{1}{k}]$ and $|S_p(r+1)| \leq \frac{1}{2}\Delta|S_p(r)|$ for $r > [\frac{1}{k}]$, we also obtain that

$$|B_p(r)| \leq 1 + \frac{(\Delta-1)^{\left[\frac{1}{k}\right]} - 1}{\Delta - 2}\Delta + \Delta(\Delta-1)^{\left[\frac{1}{k}\right]}\left(\frac{\left(\frac{\Delta}{2}\right)^{r-\left[\frac{1}{k}\right]} - 1}{\frac{\Delta}{2} - 1}\right),$$

which is much smaller than $1 + \frac{(\Delta-1)^r - 1}{\Delta-2}\Delta$. (If G is a tree and every vertex has degree $\Delta \geq 3$, then $|B_p(r)| = 1 + \frac{(\Delta-1)^r-1}{\Delta-2}\Delta$.)

Let $\mathrm{diam}(G) = \max\{d(x, y) \mid x, y \in V\}$. We obtain the following Bonnet–Myers type theorem from (1.1), which was induced immediately from Proposition 23 in [5].

Corollary 1. *Assume that $\kappa(x, y) \geq k > 0$ for any two distinct points $x, y \in V$. Then* $\mathrm{diam}(G) \leq [\frac{2}{k}]$.

We obtain Corollary 1 by considering the volume growth. The proof is more geometric and similar to the proof of the Bonnet–Myers theorem.

For a finite graph $G = G(V, E)$, a measure μ on G means that μ is a measure on the set of vertices V. Also the integration $\int_G f d\mu$ on G means $\int_V f d\mu$. Let μ_0 be the measure such that $\mu_0(A) = |A|$ for $A \subset V$. For the convenience, we use the notation that the volume of G, $\mathrm{vol}(G) = \int_G d\mu_0 = |V|$. With Corollary 1 and Theorem 1, we obtain the following corollary.

Corollary 2. *Assume that there exists $p \in V$ such that $\kappa(p, x) \geq k > 0$ for all $x \in V$. Then* $\mathrm{vol}(G) \leq C(k, [\frac{2}{k}], \Delta)$.

During the correction of the proofs, the author learned about an independent proof of Theorem 1 and Corollary 2 in "Ricci curvature of graphs" by Lin, Lu and Yau, which was recently published in Tohoku Mathematical Journal 63 (2011), 605–627. They used a modified Ollivier's Ricci curvature and their condition is slightly weaker [3].

Let $\mu * m := \int_{x\in X} d\mu(x) m_x$. (See [5].) For a finite graph, there is an invariant measure, i.e. there is a measure ν such that $\nu * m = \nu$ [1]. Precisely, since $2|E| = \sum_{x\in V} \deg(x)$, if we let

$$\nu(x) = \frac{\deg(x)}{2|E|}, \tag{1.4}$$

then ν is an invariant probability measure. Then Theorem 1 holds if we use the invariant measure ν instead of the cardinality.

Corollary 3. *Assume that there exists $p \in V$ such that $\kappa(p, x) \geq k$ for all $x \in V$, then*

$$\nu(S_p(r+1)) \leq \Delta\left(1 - \frac{k}{2}r\right)\nu(S_p(r)) \tag{1.5}$$

for $k \in \mathbb{R}$. So for $r \geq 1$,

$$\nu(S_p(r)) \leq \Delta^r \frac{\deg(p)}{2|E|} \prod_{m=0}^{r-1}\left(1 - \frac{k}{2}m\right). \tag{1.6}$$

For $C(k, r, \Delta)$ in Theorem 1,

$$\nu(B_p(r)) \leq \frac{\deg(p)}{2|E|} C(k, r, \Delta). \tag{1.7}$$

In Riemannian geometry, there are many attempts to study global structures with integral curvature conditions instead of pointwise curvature conditions [7,8,6]. (For example, the Gauss–Bonnet theorem.) In [7], Petersen and Wei obtained a Bishop–Gromov type volume comparison theorem with an integral Ricci curvature. Petersen and Sprouse obtained the Bonnet–Myers type theorem with an integral Ricci curvature [6]. We can consider the integration of Ollivier's Ricci curvature. With the integration of curvature, we can obtain the volume growth rate without pointwise positivity of curvature. Let $f_+(x) = \max\{f(x), 0\}$. Also we let $\kappa(x,x) = 0$ for integration of curvature in Theorems 2–4.

Theorem 2. *For a finite graph G, we obtain the following volume growth:*

$$|S_p(r+1)| \le \Delta\left(|S_p(r)| - \sum_{x\in S_p(r)} \frac{\kappa_+(p,x)r}{2}\right).$$

From this volume growth, we have that

$$|B_p(r)| \le \frac{\Delta^{r+1}-1}{\Delta-1} - \frac{r-1}{2}\Delta\int_{B_p(r-1)} \kappa_+(p,x)d\mu_0(x).$$

Note that $\int_{B_p(r-1)} \kappa_+(p,x)d\mu_0(x) = \sum_{x\in B_p(r-1)} \kappa_+(p,x)$. From the above theorem, we obtain the following upper bounds of diameter and volume with an integral curvature.

Theorem 3. *Let α be a positive real number. If*

$$\frac{1}{\mathrm{vol}(G)}\int_G \kappa_+(p,x)d\mu_0(x) \ge \alpha$$

for some $p \in G$, then

$$\mathrm{diam}(G) \le 2\left(\left[\frac{4}{\alpha}\right] + 2\left(\frac{1}{\alpha}-1\right)\left(1+\frac{(\Delta-1)^{\left[\frac{4}{\alpha}\right]}-1}{\Delta-2}\Delta\right)\right).$$

Let $\gamma = [\frac{4}{\alpha}] + 2(\frac{1}{\alpha}-1)(1+\frac{(\Delta-1)^{[\frac{4}{\alpha}]}-1}{\Delta-2}\Delta)$. Then

$$\mathrm{vol}(G) \le \frac{1}{1+\frac{\gamma}{2}\Delta\alpha}\frac{\Delta^{\gamma+2}-1}{\Delta-1}.$$

The upper bound of diameter in Theorem 3 is exponential in $\frac{1}{\alpha}$, which is much larger than that of Corollary 1. In Example 1, we construct a graph satisfying the condition of Theorem 3 whose diameter is exponential in $\frac{1}{\alpha}$. Hence the exponential dependency on $\frac{1}{\alpha}$ cannot be improved.

With an integration of curvature with respect to the invariant measure ν, we obtain simpler upper bounds of diameter. For the similar integral curvature as [7], we define λ_K as follows:

$$\lambda_K(x,y) = (K - \kappa(x,y))_+.$$

If $\kappa(x,y) \ge K$ everywhere, then $\lambda_K = 0$ for every x, y. Using the invariant measure ν in (1.4), we obtain the following diameter upper bounds with integrations of Ricci curvature with respect to ν.

Theorem 4. *Let ν be the invariant measure on G defined in (1.4).*

(1) *If $\int_G \kappa(x,z)d\nu(z) \ge \mathcal{K} > \frac{1}{2}$ for any $x \in G$, then*

$$\mathrm{diam}(G) \le \frac{2}{2\mathcal{K}-1}.$$

(2) *If* $\int_G \lambda_K(x,z)d\nu(z) < \epsilon < \frac{1}{2}$ *for any* $x \in G$ *and* $D = \sup_x \int_G d(x,y)d\nu(y)$, *then*

$$\operatorname{diam}(G) \le \frac{2 + 2(1-K)D}{1-2\epsilon}.$$

(3) *If* $\int_G \lambda_K(x,z)d\nu(z) < \epsilon < \frac{K}{2}$ *for any* $x \in G$, *then*

$$\operatorname{diam}(G) \le \frac{2}{K-2\epsilon}.$$

2. Preliminaries

We review some basic properties of Ollivier's Ricci curvature. Ollivier's Ricci curvature has the following properties. For an N-dimensional Riemannian manifold, let

$$dm_x^{\epsilon}(y) := \begin{cases} \dfrac{1}{\operatorname{vol}(B(x,\epsilon))} d\operatorname{vol}(y) & \text{if } y \in B_x(\epsilon) \\ 0 & \text{otherwise}. \end{cases}$$

If y is a point on the unit speed geodesic γ such that $\gamma(0) = x$ and $\gamma'(0) = v$, then

$$\kappa(x,y) = \frac{\epsilon^2 \operatorname{Ric}(v,v)}{2(N+2)} + O(\epsilon^3 + \epsilon^2 d(x,y)).$$

Hence Ollivier's Ricci curvature can be considered as an extension of Ricci curvature to a metric space [5].

In the case of a simple graph, let ξ be a matrix with terms $\xi(x',y')$ representing the mass moving from $x' \in \operatorname{supp}(m_x)$ to $y' \in \operatorname{supp}(m_y)$, where $\operatorname{supp}(\mu)$ is the support of μ. Then

$$W_1(m_x, m_y) = \inf_{\xi} \sum_{x' \sim x} \sum_{y' \sim y} d(x',y')\xi(x',y'),$$

where the infimum is taken over all matrices ξ satisfying

$$\sum_{x' \sim x} \xi(x',y') = \frac{1}{\deg(y)}, \qquad \sum_{y' \sim y} \xi(x',y') = \frac{1}{\deg(x)}.$$

From the definition, we have $\kappa(x,y) \le 1$ for any x, y. We have the following examples for graphs [2]. Assume that $x \neq y$. For a complete graph K_n with n-vertices,

$$\kappa(x,y) = \frac{n-2}{n-1}.$$

For a tree,

$$\kappa(x,y) \ge -2\left(1 - \frac{1}{\deg(x)} - \frac{1}{\deg(y)}\right)_+.$$

Let $J(x)$ be $W_1(\delta_x, m_x) = \int_X d(x,y)dm_x(y)$ [5]. Ollivier obtained the following Bonnet–Myers type theorem [5].

Theorem 5. *Suppose* $\kappa(x,y) \ge k > 0$ *for any* $x, y \in X$. *Then*

$$\operatorname{diam}(G) \le \frac{2 \sup_x J(x)}{k}.$$

If X is a graph, then $J(x) \leq 1$. If X is a simple graph, then $J(x) = 1$. Hence Corollary 1 is an immediate consequence of Theorem 5. But we will use the volume growth for the proof of Corollary 1.

The diameter bound of Theorem 3 is much larger than those of Corollary 1 or Theorem 5. This bound is exponential in $\frac{1}{\alpha}$. We will give an example for Theorem 3 such that the diameter is exponential in $\frac{1}{\alpha}$.

Example 1. Let $G = G(V, E)$ be a finite tree and $N = n! = 1 \cdot 2 \cdot 3 \cdots n$. The set of vertices

$$V = \{v_0, v_1\} \cup \bigcup_{j=1}^{N/2} V_j,$$

where $V_j = \{v_{ij} \mid 2 \leq i \leq a_j\}$, $a_j = l - 1$ if $\frac{N}{l-1} \geq j > \frac{N}{l}$ for $l = 3, \ldots, n$ and $a_j = n$ for $j \leq \frac{N}{n}$. We denote the edge connecting x and y by (xy). The set of edges is as follows:

$$E = \{(v_0 v_1)\} \cup \left\{(v_1 v_{2j}) \mid j \leq \frac{N}{2}\right\} \cup \bigcup_{j=1}^{N/2} \{(v_{ij} v_{i+1j}) \mid 2 \leq i \leq a_j - 1\}.$$

We can verify that $\kappa(v_0, v_1) = 0$ and $\kappa(v_0, v_{ij}) \geq \frac{1}{i}$. (In fact, $\kappa(v_0, v_{ij}) = \frac{1}{i}$ for $i < a_j$.) Also we obtain that for a fixed $i \leq n$, $|\{v_{i1}, v_{i2}, \ldots\}| = \frac{N}{i}$. Then for sufficiently large n,

$$\mathrm{vol}(G) = 2 + N\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \leq 2 + N \log n$$

and

$$\int_G \kappa_+(v_0, x) d\mu_0 = N\left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{n}\right)^2\right) \geq \frac{N}{2}.$$

Hence for sufficiently large n,

$$\frac{1}{\mathrm{vol}(G)} \int_G \kappa_+(v_0, x) d\mu_0 \geq \frac{1}{3 \log n}.$$

If we take $n = [e^{\frac{1}{3\alpha}}]$, then $\frac{1}{\mathrm{vol}(G)} \int_G \kappa_+(v_0, x) d\mu_0 \geq \alpha$ and $\mathrm{diam}(G) \geq n \geq e^{\frac{1}{3\alpha}}$.

3. Volume and diameter of *G*

In this section, we will prove Theorems 1–3.

Proof of Theorem 1. Let $d(p, x) = r$ and ξ be an optimal coupling of m_p and m_x. For $y' \in N(p)$, we have that $d(x', y') \geq r$ for $x' \in S_p(r+1) \cap N(x)$ and $d(x', y') \geq r - 2$ for $x' \in N(x) \setminus S_p(r+1)$. Since the supports of m_x and m_p are $N(x)$ and $N(p)$, respectively, we have

$$\begin{aligned}
(1 - \kappa(p, x))r &= (1 - \kappa(p, x))d(p, x) \geq \int d(x', y') d\xi \\
&\geq r \int_{x' \in S_p(r+1) \cap N(x)} d\xi + (r - 2) \int_{x' \in N(x) \setminus S_p(r+1)} d\xi \\
&= r \frac{|S_p(r+1) \cap N(x)|}{|N(x)|} + (r - 2) \frac{|N(x) \setminus S_p(r+1)|}{|N(x)|} \\
&= r \frac{|S_p(r+1) \cap N(x)|}{|N(x)|} + (r - 2) \frac{|N(x)| - |S_p(r+1) \cap N(x)|}{|N(x)|} \\
&= \frac{2|S_p(r+1) \cap N(x)|}{|N(x)|} + (r - 2). \qquad (3.8)
\end{aligned}$$

Note that $\deg(x) \geq |N(x)|$. (If G is simple, then $\deg(x) = |N(x)|$.) From (3.8), we obtain that

$$\begin{aligned} |S_p(r+1) \cap N(x)| &\leq \frac{2-\kappa(p,x)r}{2}|N(x)| \\ &\leq \frac{2-\kappa(p,x)r}{2}\deg(x). \end{aligned} \tag{3.9}$$

Since $S_p(r+1) = \bigcup_{x\in S_p(r)} S_p(r+1) \cap N(x)$, we obtain that

$$\begin{aligned} |S_p(r+1)| &\leq \sum_{x\in S_p(r)} |S_p(r+1) \cap N(x)| \\ &\leq \sum_{x\in S_p(r)} \frac{2-\kappa(p,x)r}{2}\deg(x). \end{aligned} \tag{3.10}$$

From $\deg(x) \leq \Delta$ and $\kappa(p,x) \geq k$, we obtain that

$$|S_p(r+1)| \leq \left(1 - \frac{k}{2}r\right)\Delta|S_p(r)|. \tag{3.11}$$

Multiplying the above inequalities from $m = 0$ to $m = r-1$, then we obtain

$$|S_p(r)| \leq \Delta^r \prod_{m=0}^{r-1}\left(1 - \frac{k}{2}m\right).$$

Since $|B_p(r)| = \sum_{n=0}^{r} |S_p(n)|$, this completes the proof of Theorem 1. □

From (3.10), we obtain Corollary 1 immediately.

Proof of Corollary 3. From (3.10) and $\nu(x) = \frac{\deg(x)}{2|E|}$, we obtain that

$$|S_p(r+1)| \leq 2|E|\frac{2-kr}{2}\nu(S_p(r)). \tag{3.12}$$

Also since $\sum_{x\in S_p(r+1)} \deg(x) \leq \Delta|S_p(r+1)|$,

$$|S_p(r+1)| \geq \frac{\sum_{x\in S_p(r+1)} \deg(x)}{\Delta} = 2|E|\frac{\nu(S_p(r+1))}{\Delta}.$$

Hence

$$\nu(S_p(r+1)) \leq \left(\frac{2-kr}{2}\Delta\right)\nu(S_p(r)).$$

Since $\nu(S_p(0)) = \frac{\deg(p)}{2|E|}$, this completes the proof of Corollary 3. □

Proof of Theorem 2. Let $A_r = \{x \in S_p(r) \mid \kappa(p,x) \geq 0\}$. From (3.9), if $x \in A_r$, then $|S_p(r+1) \cap N(x)| \leq (1 - \frac{\kappa(p,x)r}{2})\deg(x)$. If $x \notin A_r$, then $|S_p(r+1) \cap N(x)| \leq \deg(x)$. From (3.10), we obtain that

$$\begin{aligned} |S_p(r+1)| &\leq \sum_{x\in S_p(r)\setminus A_r} \Delta + \sum_{x\in A_r}\left(1 - \frac{\kappa(p,x)r}{2}\right)\Delta \\ &= \sum_{x\in S_p(r)} \Delta - \sum_{x\in A_r} \frac{\kappa(p,x)r}{2}\Delta \end{aligned}$$

$$= \Delta \left(|S_p(r)| - \sum_{x \in A_r} \frac{\kappa(p,x)r}{2} \right)$$

$$= \Delta \left(|S_p(r)| - \sum_{x \in S_p(r)} \frac{\kappa_+(p,x)r}{2} \right). \tag{3.13}$$

Then

$$|S_p(2)| \le \Delta \left(|S_p(1)| - \sum_{x \in S_p(1)} \frac{\kappa_+(p,x)}{2} \right)$$

$$\begin{aligned} |S_p(3)| &\le \Delta \left(|S_p(2)| - \sum_{x \in S_p(2)} \frac{\kappa_+(p,x)}{2} \cdot 2 \right) \\ &\le \Delta^2 |S_p(1)| - \left(\Delta^2 \sum_{x \in S_p(1)} \frac{\kappa_+(p,x)}{2} + \Delta \sum_{x \in S_p(2)} \frac{\kappa_+(p,x)}{2} \cdot 2 \right) \\ &\le \Delta^2 |S_p(1)| - \Delta \left(\sum_{x \in S_p(1)} \frac{\kappa_+(p,x)}{2} + \sum_{x \in S_p(2)} \frac{\kappa_+(p,x)}{2} \cdot 2 \right). \end{aligned} \tag{3.14}$$

Since $r - 1 \le k(r-k)$ for $1 \le k \le r-1$ and $|S_p(1)| \le \Delta$,

$$\begin{aligned} |B_p(r)| &\le \sum_{a \le r} |S_p(a)| \\ &\le \left(1 + \Delta + \Delta^2 + \cdots + \Delta^r\right) \\ &\quad - \Delta \left(\sum_{x \in S_p(1)} \frac{\kappa_+(p,x)}{2} \cdot (r-1) + \sum_{x \in S_p(2)} \frac{\kappa_+(p,x)}{2} \cdot 2(r-2) \right. \\ &\quad \left. + \sum_{x \in S_p(3)} \frac{\kappa_+(p,x)}{2} \cdot 3(r-3) + \cdots + \sum_{x \in S_p(r-1)} \frac{\kappa_+(p,x)}{2} \cdot (r-1) \right) \\ &\le \frac{\Delta^{r+1} - 1}{\Delta - 1} - \frac{r-1}{2} \Delta \int_{B_p(r-1)} \kappa_+(p,x) d\mu_0(x). \quad \square \end{aligned} \tag{3.15}$$

Proof of Theorem 3. Fix a vertex p of G. We assume that the set of vertices $V = B_p(R)$ and $B_p(R) \setminus B_p(R-1) \ne \emptyset$ for some $R > 0$. The inequality $\int_{B_p(R)} \kappa_+(p,x) d\mu_0(x) \ge \alpha |B_p(R)|$ means

$$\sum_{r=0}^{R} \sum_{x \in S_p(r)} \kappa_+(p,x) \ge \alpha \sum_{r=0}^{R} |S_p(r)|. \tag{3.16}$$

If for any r such that $r_0 < r \le R$,

$$\frac{\sum_{x \in S_p(r)} \kappa_+(p,x)}{|S_p(r)|} < \frac{\alpha}{2},$$

then

$$\frac{\int_{B_p(R)} \kappa_+(p,x)d\mu_0}{|B_p(R)|} = \frac{\sum_{x\in S_p(0)} \kappa_+(p,x) + \cdots + \sum_{x\in S_p(R)} \kappa_+(p,x)}{|S_p(0)| + \cdots + |S_p(R)|}$$
$$< \frac{|S_p(0)| + \cdots + |S_p(r_0)| + \frac{\alpha}{2}(|S_p(r_0+1)| + \cdots + |S_p(R)|)}{|S_p(0)| + \cdots + |S_p(r_0)| + (|S_p(r_0+1)| + \cdots + |S_p(R)|)} \tag{3.17}$$

from $\kappa(p,x) \leq 1$. Let $X = |S_p(0)| + \cdots + |S_p(r_0)|$ and $Y = |S_p(r_0+1)| + \cdots + |S_p(R)|$. Then we have

$$X \leq 1 + \frac{(\Delta-1)^{r_0} - 1}{\Delta - 2}\Delta$$

and

$$Y \geq R - r_0.$$

If

$$R \geq r_0 + 2\left(\frac{1}{\alpha} - 1\right)\left(1 + \frac{(\Delta-1)^{r_0} - 1}{\Delta-2}\Delta\right),$$

then $Y \geq 2(\frac{1}{\alpha} - 1)X$. Then

$$\begin{aligned}\frac{\int_{B_p(R)} \kappa_+(p,x)d\mu_0}{|B_p(R)|} &< \frac{X + \frac{\alpha}{2}Y}{X+Y} \\ &\leq \frac{\alpha}{2} + \frac{\left(1-\frac{\alpha}{2}\right)X}{\left(1+2\left(\frac{1}{\alpha}-1\right)\right)X} \\ &= \alpha,\end{aligned} \tag{3.18}$$

which is a contradiction to (3.16). So if we assume that R is larger than $2(\frac{1}{\alpha}-1)(1+\frac{(\Delta-1)^{[\frac{4}{\alpha}]}-1}{\Delta-2}\Delta)+[\frac{4}{\alpha}]$ and we choose r_0 to be $[\frac{4}{\alpha}]$, then there exists r_1 such that $r_0 < r_1 \leq R$ and

$$\frac{\sum_{x\in S_p(r_1)} \kappa_+(p,x)}{|S_p(r_1)|} \geq \frac{\alpha}{2}. \tag{3.19}$$

By Theorem 2 and (3.19), if $|S_p(r_1)| \neq 0$, then

$$\begin{aligned}|S_p(r_1+1)| &\leq \Delta\left(|S_p(r_1)| - \sum_{x\in S_p(r_1)} \frac{\kappa_+(p,x)r_1}{2}\right) \\ &< \Delta\left(|S_p(r_1)| - \frac{2}{\alpha}\sum_{x\in S_p(r_1)} \kappa_+(p,x)\right) \leq 0\end{aligned}$$

from $r_1 \geq r_0 + 1 = [\frac{4}{\alpha}] + 1 > \frac{4}{\alpha}$, which is a contradiction from $|S_p(r)| \geq 0$. If $|S_p(r_1)| = 0$ for $r_1 \leq R$, then $S_p(R) = B_p(R) \setminus B_p(R-1) = \emptyset$, which is a contradiction to our assumption. Hence we obtain that

$$R \leq \left[\frac{4}{\alpha}\right] + 2\left(\frac{1}{\alpha} - 1\right)\left(1 + \frac{(\Delta-1)^{\left[\frac{4}{\alpha}\right]} - 1}{\Delta - 2}\Delta\right).$$

Then $\mathrm{diam}(G) \leq 2R \leq 2([\frac{4}{\alpha}] + 2(\frac{1}{\alpha} - 1)(1 + \frac{(\Delta-1)^{[\frac{4}{\alpha}]}-1}{\Delta-2}\Delta))$.

For the volume of G, let $r = \gamma + 1$ in Theorem 2. Then the set of vertices V satisfies that $V = B_p(\gamma + 1) = B_p(\gamma)$. So we have

$$|V| \leq \frac{\Delta^{\gamma+2} - 1}{\Delta - 1} - \frac{\gamma}{2}\Delta \int_G \kappa_+(p, x) d\mu_0(x) \\ \leq \frac{\Delta^{\gamma+2} - 1}{\Delta - 1} - \frac{\gamma}{2}\Delta\alpha|V|. \tag{3.20}$$

Then we obtain $|V| \leq \frac{1}{1+\frac{\gamma}{2}\Delta\alpha} \frac{\Delta^{\gamma+2}-1}{\Delta-1}$, which completes the proof of Theorem 3. □

4. Invariant measure, integral curvature and diameter

In this section, we will prove Theorem 4.

Recall that $\lambda_K(x, y) = (K - \kappa(x, y))_+$ and δ_x is the Dirac measure. Similarly as [5], we will prove the following lemma.

Lemma 1. (1) *Let* (X, d, m) *be a metric space with a random walk* m*. Let* $\operatorname{diam}(X) < \infty$ *and* ν *be an invariant measure. Then we have*

$$W_1(m_x, \nu) \leq \operatorname{diam}(X) \int (1 - \kappa(x, z)) d\nu(z).$$

(2) *If* $\int \lambda_K(x, z) d\nu(z) < \epsilon$*, then*

$$W_1(m_x, \nu) \leq (1 - K) W_1(\delta_x, \nu) + \epsilon \operatorname{diam}(X).$$

Proof. (1) For measures μ_1, μ_2, let Ξ be a coupling witnessing $W_1(\mu_1, \mu_2)$ and ξ_{xy} be an optimal coupling between m_x and m_y. Then $\int_{X\times X} d\Xi(x, y)\xi_{xy}$ is a coupling between $\mu_1 * m$ and $\mu_2 * m$. Then

$$W_1(\mu_1 * m, \mu_2 * m) \leq \int_{x,y} d(x, y) d\left(\int_{x',y'} d\Xi(x', y')\xi_{x'y'}\right)(x, y) \\ = \int_{x,y,x',y'} d\Xi(x', y') d\xi_{x'y'}(x, y) d(x, y) \\ \leq \int_{x',y'} d\Xi(x', y')(1 - \kappa(x', y')) d(x', y'). \tag{4.21}$$

If we let $\mu_1 = \delta_x$ and $\mu_2 = \nu$, then $\mu_1 * m = m_x$ and $\mu_2 * m = \nu$. The only coupling between δ_x and a measure μ is $\delta_x \times \mu$, i.e. $\delta_x \times \mu(U) = \delta(\pi_1(U))\mu(\pi_2(U))$ for $U \subset X \times X$ and $\pi_1(x, y) = x$, $\pi_2(x, y) = y$. Hence

$$W_1(m_x, \nu) \leq \operatorname{diam}(X) \int_{y'} (1 - \kappa(x, y')) d\nu(y'). \tag{4.22}$$

(2) By the same arguments as above, we obtain that

$$W_1(m_x, \nu) \leq \int_{x',y'} d\Xi(x', y')(1 - K + K - \kappa(x', y')) d(x', y') \\ \leq (1 - K) W_1(\delta_x, \nu) + \operatorname{diam}(X) \int_{y'} \lambda_K(x, y') d\nu(y') \\ \leq (1 - K) W_1(\delta_x, \nu) + \epsilon \operatorname{diam}(X). \quad \square \tag{4.23}$$

Proof of Theorem 4. (1) Recall that $J(x) = W_1(\delta_x, m_x) = \int_G d(x, y)dm_x(y) \le 1$. We take x, y such that $d(x, y) = \mathrm{diam}(G)$. By Lemma 1(1), we have

$$\begin{aligned} d(x, y) &\le W_1(\delta_x, \nu) + W_1(\delta_y, \nu) \\ &\le W_1(\delta_x, m_x) + W_1(m_x, \nu) + W_1(m_y, \delta_y) + W_1(m_y, \nu) \\ &\le J(x) + J(y) + W_1(m_x, \nu) + W_1(m_y, \nu) \\ &\le 2 + \mathrm{diam}(G)\left(\int (1 - \kappa(x, z))d\nu(z) + \int (1 - \kappa(y, z))d\nu(z)\right). \end{aligned} \tag{4.24}$$

Since $\mathcal{K} \le \inf_x \int_G \kappa(x, z)d\nu$, we have

$$\begin{aligned} \mathrm{diam}(G) &\le 2 + \mathrm{diam}(G)\left(\int (1 - \kappa(x, z))d\nu(z) + \int (1 - \kappa(y, z))d\nu(z)\right) \\ &\le 2 + 2\mathrm{diam}(G)(1 - \mathcal{K}). \end{aligned} \tag{4.25}$$

So we obtain that

$$\mathrm{diam}(G) \le \frac{2}{2\mathcal{K} - 1}. \tag{4.26}$$

(2) By Lemma 1(2), we have

$$W_1(m_x, \nu) \le (1 - K)W_1(\delta_x, \nu) + \epsilon\mathrm{diam}(X). \tag{4.27}$$

Since $\sup_x \int \lambda_K(x, z)d\nu(z) < \epsilon$ and $W_1(\delta_x, \nu) = \int_G d(x, y)d\nu(y)$, similarly as (4.24),

$$\begin{aligned} d(x, y) &\le J(x) + J(y) + W_1(m_x, \nu) + W_1(m_y, \nu) \\ &\le 2 + 2(1 - K)D + 2\epsilon\mathrm{diam}(G). \end{aligned} \tag{4.28}$$

Then we also have

$$\mathrm{diam}(G) \le \frac{2 + 2(1 - K)D}{1 - 2\epsilon}. \tag{4.29}$$

(3) We have

$$\begin{aligned} W_1(m_x, \nu) &\ge W_1(\delta_x, \nu) - W_1(\delta_x, m_x) \\ &= W_1(\delta_x, \nu) - J(x). \end{aligned}$$

By (4.23) and the above inequality, we have

$$W_1(\delta_x, \nu) - J(x) \le W_1(m_x, \nu) \le (1 - K)W_1(\delta_x, \nu) + \epsilon\mathrm{diam}(G), \tag{4.30}$$

which implies that

$$W_1(\delta_x, \nu) \le \frac{J(x) + \epsilon\mathrm{diam}(G)}{K}. \tag{4.31}$$

Since $D = \sup_x \int d(x, y)d\nu(y) = \sup_x W_1(\delta_x, \nu)$, we have

$$D \le \frac{\sup_x J(x) + \epsilon\mathrm{diam}(G)}{K} \le \frac{1 + \epsilon\mathrm{diam}(G)}{K}. \tag{4.32}$$

With Theorem 4(2), we obtain that

$$\begin{aligned}\operatorname{diam}(G) &\leq \frac{2}{1-2\epsilon}+\frac{2(1-K)}{1-2\epsilon}\left(\frac{1+\epsilon\operatorname{diam}(G)}{K}\right)\\ &= \frac{2}{1-2\epsilon}+\frac{2(1-K)}{(1-2\epsilon)K}+\frac{2(1-K)\epsilon}{(1-2\epsilon)K}\operatorname{diam}(G).\end{aligned}\tag{4.33}$$

If $\epsilon < \frac{K}{2}$, then $\frac{2(1-K)\epsilon}{(1-2\epsilon)K} < 1$. Then

$$\begin{aligned}\operatorname{diam}(G) &\leq \frac{(1-2\epsilon)K}{K-2\epsilon}\left(\frac{2}{1-2\epsilon}+\frac{2(1-K)}{(1-2\epsilon)K}\right)\\ &= \frac{2}{K-2\epsilon},\end{aligned}\tag{4.34}$$

which completes the proof of Theorem 4(3). □

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